Section 3.1.

4. $f(x,y) = e^{-xy^2} + y^3 x^4$

$\frac{\partial f}{\partial x} = -y^2 e^{-xy^2} + 4y^3x^3$, $\frac{\partial f}{\partial y} = -2xye^{-xy^2} + 3y^2x^4$.

$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = 2xy^3 e^{-xy^2} + 12y^2x^3$,

$\frac{\partial^2 f}{\partial x^2} = y^4 e^{-xy^2} + 12y^3x^2$, $\frac{\partial^2 f}{\partial y^2} = -2xe^{-xy^2} + 4x^2y^2e^{-xy^2} + 6yx^4$.

Rubric: 1pt for each derivative.

22. $\begin{cases} x = u+v \\ y = u-v \end{cases}$, show that $\frac{\partial^2 w}{\partial u \partial v} = \frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2}$.

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial v} = \frac{\partial w}{\partial x} - \frac{\partial w}{\partial y}$$

$$\frac{\partial^2 w}{\partial u \partial v} = \frac{\partial}{\partial u}\left(\frac{\partial w}{\partial v}\right) = \frac{\partial}{\partial u}\left(\frac{\partial w}{\partial x} - \frac{\partial w}{\partial y}\right) = \frac{\partial^2 w}{\partial x^2}\cdot\frac{\partial x}{\partial u} + \frac{\partial^2 w}{\partial x \partial y}\cdot\frac{\partial y}{\partial u} - \frac{\partial^2 w}{\partial y \partial x}\cdot\frac{\partial x}{\partial u} - \frac{\partial^2 w}{\partial y^2}\frac{\partial y}{\partial u}$$

$$= \frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2}.$$

Rubric: 2pts for using chain rule right,
3pts for $\frac{\partial w}{\partial v}$ or $\frac{\partial w}{\partial u}$ and $\frac{\partial^2 w}{\partial u \partial v}$.

Section 3.2.

2. (a) $ax + by$

(b) $ax + by$

(c) Same, since Taylor gives best polynomial approx.

Rubric: 2pts for (a)/(b), 1pt for (c).

Section 3.3.

6. $f(x,y) = x^2 - 3xy + 5x - 2y + 6y^2 + 8$.

$\nabla f = (2x - 3y + 5, -3x - 2 + 12y) = 0$

$\Rightarrow \begin{cases} x = -\frac{18}{5} \\ y = -\frac{11}{15} \end{cases}$, $\frac{\partial^2 f}{\partial x^2} = 2$, $\frac{\partial^2 f}{\partial y^2} = 12$, $\frac{\partial^2 f}{\partial x \partial y} = -3$

$D = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = \cancel{6 - 12^2 < 0}$

$24 - (-3)^2 > 0$

So f has a local min at $\left(-\frac{18}{5}, -\frac{11}{15}\right)$.

Rubric: 2 pts for finding critical pt,

3pts for determine type.

18 (a). $f(x,y,z) = x^2 + y^2 + z^2 + kyz$, $\nabla f|_0 = (0,0,0)$

So 0 is a critical pt.

(b). Hessian of $f = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & k \\ 0 & k & 2 \end{bmatrix}$

Rubric:
2pts for (a)
3pts for (b)

So f has a local min at $(0,0,0)$ if $\det \begin{bmatrix} 2 & k \\ k & 2 \end{bmatrix} > 0$

$\Rightarrow 4 - k^2 > 0$, $-2 < k < 2$.

26. $f(x,y) = ax^2 + by^2$,

(a). $\nabla f = (2ax, 2by) = 0 \Rightarrow \begin{cases} x = 0 \\ y = 0 \end{cases}$

(b). Hessian of $f = \begin{bmatrix} 2a & 0 \\ 0 & 2b \end{bmatrix}$, $a, b \neq 0$.

So If $a, b > 0$, local min

If $a, b < 0$, local max.

If $a \cdot b < 0$, saddle pt.

Rubric: 2pts for (a) 3pts for (b).